

On Inverse Spectral Problems for Second Order Integro-differential Operators

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Abstract. Inverse spectral problems are studied for the second order integro-differential operators on a finite interval. Properties of spectral characteristic are established, and the uniqueness theorem is proved for this class of inverse problems.

AMS Classification: 47G20 45J05 44A15

Key words: integro-differential operators, inverse spectral problems, uniqueness theorem

1. Inverse spectral problems consist in recovering operators from their spectral characteristics. Such problems often appear in mathematics, mechanics, physics, electronics, geophysics and other branches of natural sciences and engineering. The greatest success in the inverse problem theory has been achieved for the Sturm-Liouville operator (see, e.g., [1-3]) and afterwards for higher-order differential operators [4-6] and other classes of differential operators.

For integro-differential and other classes of nonlocal operators inverse problems are more difficult for investigation, and the main classical methods (transformation operator method and the method of spectral mappings [1-6]) either are not applicable to them or require essential modifications, and for such operators the general inverse problem theory does not exist. At the same time, nonlocal and, in particular, integro-differential operators are of great interest, because they have many applications (see, e.g., [7]). We note that some aspects of inverse problems for integro-differential operators were studied in [8-10] and other works. In the present paper we study inverse spectral problem for one class of second order integro-differential operators on a finite interval. Properties of spectral characteristic are established, and the uniqueness theorem is proved for this class of inverse problems.

2. Consider the integro-differential equation

$$\ell y := -y''(x) + q(x)y(x) + \int_0^x M(x,t)y(t) dt = \lambda y(x), \quad x \in [0, \pi], \quad (1)$$

where $q(x)$ and $M(x,t)$ are integrable complex-valued functions. Let $C(x, \lambda)$ and $S(x, \lambda)$ be solutions of Eq. (1) with the initial conditions

$$C(0, \lambda) = S'(0, \lambda) = 1, \quad C'(0, \lambda) = S(0, \lambda) = 0.$$

For each fixed $x \in [0, \pi]$, the functions $C^{(\nu)}(x, \lambda)$ and $S^{(\nu)}(x, \lambda)$, $\nu = 0, 1$, are entire in λ of order $1/2$. Denote $\Delta_1(\lambda) := S(\pi, \lambda)$, $\Delta_2(\lambda) := C(\pi, \lambda)$. Zeros $\{\lambda_{nk}\}_{n \geq 1}$ of the entire function $\Delta_k(\lambda)$ coincide with the eigenvalues of the boundary value problem $L_k = L_k(M, q)$ for Eq. (1) with the conditions $y^{(k-1)}(0) = y(\pi) = 0$. The function $\Delta_k(\lambda)$ is called the characteristic function for L_k .

Let $\Phi(x, \lambda)$ be the solution of Eq. (1) under the conditions $\Phi(0, \lambda) = 1$, $\Phi(\pi, \lambda) = 0$. Denote $N(\lambda) := \Phi'(0, \lambda)$. Then

$$\Phi(x, \lambda) = C(x, \lambda) + N(\lambda)S(x, \lambda), \quad N(\lambda) = -\Delta_2(\lambda)/\Delta_1(\lambda). \quad (2)$$

The function $N(\lambda)$ is called the Weyl-type function for ℓ . It follows from (2) that the function $N(\lambda)$ is meromorphic in λ with poles $\{\lambda_{n1}\}_{n \geq 1}$ and zeros $\{\lambda_{n2}\}_{n \geq 1}$. Let $M(x, t)$ be known a priori. The inverse problem is formulated as follows.

Inverse problem 1. Given $N(\lambda)$, construct $q(x)$.

This inverse problem is an analogue of the classical inverse problem of recovering the Sturm-Liouville operator from the given Weyl function [3].

3. Let $\lambda = \rho^2$. By the well-known method (see, e.g., [3]) we have for $|\rho| \rightarrow \infty$:

$$S(x, \lambda) = \frac{\sin \rho x}{\rho} - \frac{\cos \rho x}{2\rho^2} \int_0^x q(t) dt + o\left(\frac{1}{\rho}\right), \quad C(x, \lambda) = \cos \rho x + \frac{\sin \rho x}{2\rho} \int_0^x q(t) dt + o\left(\frac{1}{\rho}\right),$$

and consequently,

$$\Delta_1(\lambda) = \frac{\sin \rho \pi}{\rho} - \frac{\omega \cos \rho \pi}{2\rho^2} + o\left(\frac{1}{\rho}\right), \quad \Delta_2(\lambda) = \cos \rho \pi + \frac{\omega \sin \rho \pi}{2\rho} + o\left(\frac{1}{\rho}\right), \quad (3)$$

where $\omega := \int_0^\pi q(t) dt$. Using (3), by standard calculations [3] we obtain

$$\rho_{n1} := \sqrt{\lambda_{n1}} = n + \frac{\omega}{2n} + o\left(\frac{1}{n}\right), \quad \rho_{n2} := \sqrt{\lambda_{n2}} = n - \frac{1}{2} + \frac{\omega}{2n} + o\left(\frac{1}{n}\right).$$

Moreover, the specification of $\{\lambda_{nk}\}_{n \geq 1}$ uniquely determines the characteristic function $\Delta_k(\lambda)$ by the formulae [3]:

$$\Delta_1(\lambda) = \pi \prod_{n=1}^{\infty} \frac{\lambda_{n1} - \lambda}{n^2}, \quad \Delta_2(\lambda) = \prod_{n=1}^{\infty} \frac{\lambda_{n2} - \lambda}{(n - 1/2)^2}. \quad (4)$$

Taking (2) and (4) into account we conclude that Inverse problem 1 is equivalent to the following Borg-type inverse problem.

Inverse problem 2. *Given two spectra $\{\lambda_{nk}\}_{n \geq 1}$, $k = 1, 2$, construct $q(x)$.*

Denote

$$\ell^* z := -z''(x) + q(x)z(x) + \int_x^\pi M(t, x)z(t) dt.$$

Obviously,

$$\int_0^\pi \ell y(x) \cdot z(x) dx = \left|_0^\pi (yz' - y'z) + \int_0^\pi y(x) \cdot \ell^* z(x) dx. \quad (5)$$

If $y(x, \lambda)$ and $z(x, \mu)$ are solutions of the equations $\ell y = \lambda y$ and $\ell^* z = \mu z$, respectively, then (5) yields

$$(\lambda - \mu) \int_0^\pi y(x, \lambda) z(x, \mu) dx = \left|_0^\pi (yz' - y'z). \quad (6)$$

Let $C_*(x, \lambda)$ and $S_*(x, \lambda)$ be solutions of the equation

$$\ell^* z = \lambda z \quad (7)$$

with the initial conditions

$$C_*(\pi, \lambda) = -S'_*(\pi, \lambda) = 1, \quad C'_*(\pi, \lambda) = S_*(\pi, \lambda) = 0.$$

Denote $\Delta_1^*(\lambda) := S_*(0, \lambda)$, $\Delta_2^*(\lambda) := -S'_*(0, \lambda)$. It follows from (6) with $\mu = \lambda$ that

$$S(\pi, \lambda) \equiv S_*(0, \lambda), \quad S'(\pi, \lambda) \equiv C_*(0, \lambda), \quad C(\pi, \lambda) \equiv -S'_*(0, \lambda), \quad C'(\pi, \lambda) \equiv -C_*(0, \lambda), \quad (8)$$

hence,

$$\Delta_1^*(\lambda) \equiv \Delta_1(\lambda), \quad \Delta_2^*(\lambda) \equiv \Delta_2(\lambda).$$

Let $\Phi_*(x, \lambda)$ be the solution of Eq. (7) under the conditions $\Phi_*(0, \lambda) = 1$, $\Phi_*(\pi, \lambda) = 0$. Denote $N^*(\lambda) := \Phi'_*(0, \lambda)$. Then

$$\Phi_*(x, \lambda) = \frac{S_*(x, \lambda)}{S_*(0, \lambda)}, \quad N^*(\lambda) = \frac{S'_*(0, \lambda)}{S_*(0, \lambda)}.$$

Together with (8) this yields $N^*(\lambda) \equiv N(\lambda)$.

It is known (see, e.g., [11]) that there exists a fundamental system of solutions $\{y_1(x, \rho), y_2(x, \rho)\}$, $\text{Im } \rho \geq 0$, $x \in [0, \pi]$ for Eq. (1) such that for $|\rho| \rightarrow \infty$, $\nu = 0, 1$:

$$y_1^{(\nu)}(x, \rho) = (i\rho)^\nu \exp(i\rho x) + O(\rho^{-1}), \quad y_2^{(\nu)}(x, \rho) = (-i\rho)^\nu \exp(-i\rho x)(1 + O(\rho^{-1})).$$

Similarly, there exists a fundamental system of solutions $\{z_1(x, \rho), z_2(x, \rho)\}$, $\text{Im } \rho \geq 0$, $x \in [0, \pi]$ for Eq. (7) such that for $|\rho| \rightarrow \infty$, $\nu = 0, 1$:

$$z_1^{(\nu)}(x, \rho) = (i\rho)^\nu \exp(i\rho x)(1 + O(\rho^{-1})), \quad z_2^{(\nu)}(x, \rho) = (-i\rho)^\nu \exp(-i\rho x) + O(\rho^{-1} \exp(-i\rho\pi)).$$

Fix $\delta, \varepsilon \in (0, \pi/2)$. Denote $Q := \{\rho : \arg \rho \in [\delta, \pi - \delta]\}$. Using these fundamental systems of solutions we obtain the following asymptotics for $\rho \in Q$, $|\rho| \rightarrow \infty$, uniformly in $x \in [0, \pi - \varepsilon]$:

$$\Phi(x, \lambda) = \exp(i\rho x)(1 + O(\rho^{-1})) + O(\rho^{-1}), \quad \Phi_*(x, \lambda) = \exp(i\rho x)(1 + O(\rho^{-1})). \quad (9)$$

4. In this section we provide an algorithm for the solution of Inverse problem 1. For this purpose together with L_k we consider the boundary value problems $\tilde{L}_k := L_k(M, \tilde{q})$ of the same form but with a different potential $\tilde{q}(x)$. We agree that everywhere below if a certain symbol α denotes an object related to L_k , then $\tilde{\alpha}$ will denote the analogous object related to \tilde{L}_k , and $\hat{\alpha} := \alpha - \tilde{\alpha}$.

Lemma 1. *Let $M(x, t) \equiv \tilde{M}(x, t)$. Then*

$$\int_0^\pi \hat{q}(x) \Phi(x, \lambda) \tilde{\Phi}_*(x, \lambda) dx \equiv -\hat{N}(\lambda), \quad (10)$$

where $\hat{q}(x) = q(x) - \tilde{q}(x)$, $\hat{N}(\lambda) = N(\lambda) - \tilde{N}(\lambda)$.

Proof. One has

$$\begin{aligned} -\Phi''(x, \lambda) + q(x)\Phi(x, \lambda) + \int_0^x M(x, t)\Phi(t, \lambda) dt &= \lambda\Phi(x, \lambda), \\ -\tilde{\Phi}_*''(x, \lambda) + \tilde{q}(x)\tilde{\Phi}_*(x, \lambda) + \int_x^\pi M(t, x)\tilde{\Phi}_*(t, \lambda) dt &= \lambda\tilde{\Phi}_*(x, \lambda). \end{aligned}$$

We multiply the first relation by $\tilde{\Phi}_*(x, \lambda)$, then subtract the second relation multiplying by $\Phi(x, \lambda)$ and integrate with respect to x :

$$\begin{aligned} \int_0^\pi \hat{q}(x) \Phi(x, \lambda) \tilde{\Phi}_*(x, \lambda) dx + \int_0^\pi \tilde{\Phi}_*(x, \lambda) dx \int_0^x M(x, t)\Phi(t, \lambda) dt \\ - \int_0^\pi \Phi(x, \lambda) dx \int_x^\pi M(t, x)\tilde{\Phi}_*(t, \lambda) dt = \int_0^\pi \left(\Phi'(x, \lambda) \tilde{\Phi}_*(x, \lambda) - \Phi(x, \lambda) \tilde{\Phi}_*'(x, \lambda) \right) dx. \end{aligned}$$

Taking the relations $\Phi(0, \lambda) = 1$, $\Phi(\pi, \lambda) = 0$, $\tilde{\Phi}_*(0, \lambda) = 1$, $\tilde{\Phi}_*(\pi, \lambda) = 0$, $N(\lambda) = \Phi'(0, \lambda)$, $\tilde{N}^*(\lambda) = \tilde{\Phi}_*'(0, \lambda)$ and $\tilde{N}^*(\lambda) = \tilde{N}(\lambda)$ into account we arrive at (10).

Lemma 2. *Let*

$$r(x) = \frac{x^k}{k!} \left(\gamma + p(x) \right), \quad H(x, \rho) = \exp(2i\rho x) \left(1 + \frac{\xi(x, \rho)}{\rho} \right) + \exp(i\rho x) \frac{\eta(x, \rho)}{\rho}, \quad x \in [0, \pi],$$

where $p(x) \in C[0, \pi]$, $p(0) = 0$, and where the functions $\xi(x, \rho), \eta(x, \rho)$ are continuous and bounded for $x \in [0, \pi]$, $\rho \in Q$, $|\rho| \geq \rho^*$. Then for $|\rho| \rightarrow \infty$, $\rho \in Q$,

$$\int_0^\pi r(x) H(x, \rho) dx = \frac{1}{(-2i\rho)^{k+1}} (\gamma + o(1)).$$

Proof. We calculate

$$(-2i\rho)^{k+1} \int_0^\pi r(x)H(x, \rho) dx = I_1(\rho) + I_2(\rho) + I_3(\rho) + I_4(\rho),$$

where

$$\begin{aligned} I_1(\rho) &= \gamma(-2i\rho)^{k+1} \int_0^\pi \frac{x^k}{k!} \exp(2i\rho x) dx, \\ I_2(\rho) &= (-2i\rho)^{k+1} \int_0^\pi \frac{x^k}{k!} p(x) \exp(2i\rho x) dx, \\ I_3(\rho) &= (-2i)^{k+1} \rho^k \int_0^\pi r(x) \exp(2i\rho x) \xi(x, \rho) dx, \\ I_4(\rho) &= (-2i)^{k+1} \rho^k \int_0^\pi r(x) \exp(i\rho x) \eta(x, \rho) dx, \end{aligned}$$

Since

$$\int_0^\infty \frac{x^k}{k!} \exp(2i\rho x) dx = \frac{1}{(-2i\rho)^{k+1}}, \quad \rho \in Q,$$

it follows that $I_1(\rho) - \gamma \rightarrow 0$ as $|\rho| \rightarrow \infty$, $\rho \in Q$. If $\rho \in Q$, then there exists $\varepsilon_0 > 0$ such that

$$|Im \rho| \geq \varepsilon_0 |\rho| \quad \text{for } \rho \in Q. \quad (11)$$

Take $\varepsilon > 0$ and choose $\delta = \delta(\varepsilon)$ such that for $x \in [0, \delta]$, $|p(x)| < \frac{\varepsilon}{2} \varepsilon_0^{k+1}$, where ε_0 is defined in (11). Then, using (11) we infer

$$\begin{aligned} |I_2(\rho)| &< \frac{\varepsilon}{2} (2|\rho|\varepsilon_0)^{k+1} \int_0^\delta \frac{x^k}{k!} \exp(-2\varepsilon_0|\rho|x) dx + (2|\rho|)^{k+1} \int_\delta^\pi \frac{x^k}{k!} |p(x)| \exp(-2\varepsilon_0|\rho|x) dx \\ &< \frac{\varepsilon}{2} + (2|\rho|)^{k+1} \exp(-2\varepsilon_0|\rho|\delta) \int_0^{\pi-\delta} \frac{(x+\delta)^k}{k!} |p(x+\delta)| \exp(-2\varepsilon_0|\rho|x) dx. \end{aligned}$$

By arbitrariness of ε we obtain that $I_2(\rho) \rightarrow 0$ for $|\rho| \rightarrow \infty$, $\rho \in Q$.

Since $|(\gamma + p(x))\xi(x, \rho)| < C$, then for $|\rho| \rightarrow \infty$, $\rho \in Q$,

$$|I_3(\rho)| < C|\rho|^k \int_0^\pi \frac{x^k}{k!} \exp(-2\varepsilon_0|\rho|x) dx \leq \frac{C}{|\rho|\varepsilon_0^{k+1}},$$

hence $I_3(\rho) \rightarrow 0$ for $|\rho| \rightarrow \infty$, $\rho \in Q$. Similarly, one gets $I_4(\rho) \rightarrow 0$ for $|\rho| \rightarrow \infty$, $\rho \in Q$. \square

For simplicity, we will assume that $q(x)$ is analytic on $[0, \pi]$. Suppose that for a certain fixed $k \geq 0$ the Taylor coefficients $q_j := q^{(j)}(0)$, $j = \overline{0, k-1}$, have been already found. Let us choose a model potential $\tilde{q}(x)$ such that the first k Taylor coefficients of q and \tilde{q} coincide, i.e. $\tilde{q}_j = q_j$, $j = \overline{0, k-1}$. Then, using (9)-(10) and Lemma 2, we can calculate the next Taylor coefficient $q_k = q^{(k)}(0)$. Namely, the following assertion is valid.

Lemma 3. *Fix k . Let the functions $q(x)$ and $\tilde{q}(x)$ be analytic for $x \in [0, \pi]$, with $\hat{q}_j := q_j - \tilde{q}_j = 0$ for $j = \overline{0, k-1}$. Then*

$$\hat{q}_k = - \lim_{\substack{|\rho| \rightarrow \infty \\ \rho \in Q}} (-2i\rho)^{k+1} \hat{N}(\lambda). \quad (12)$$

Thus, we arrive at the following algorithm for the solution of Inverse Problem 1.

Algorithm 1. *Let the Weyl-type function $N(\lambda)$ be given. Then:*

(i) We calculate $q_k = q^{(k)}(0)$, $k \geq 0$. For this purpose we successively perform the following operations for $k = 0, 1, 2, \dots$: We construct a model potential $\tilde{q}(x)$ such that $\tilde{q}_j = q_j$, $j = \overline{0, k-1}$ and arbitrary in the rest, and we calculate $q_k = q^{(k)}(0)$ by (12).

(ii) We construct the function $q(x)$ by the formula

$$q(x) = \sum_{k=0}^{\infty} q_k \frac{x^k}{k!}, \quad 0 < x < R,$$

where

$$R = \left(\overline{\lim_{k \rightarrow \infty}} \left(\frac{|q_k|}{k!} \right)^{1/k} \right)^{-1}.$$

If $R < \pi$ then for $R < x < \pi$ the function $q(x)$ is constructed by analytic continuation.

Acknowledgment. This work was supported in part by Grant 1.1660.2017/PCh of the Russian Ministry of Education and Science and by Grants 16-01-00015, 17-51-53180 of Russian Foundation for Basic Research.

REFERENCES

- [1] Marchenko V.A., Sturm-Liouville operators and their applications. "Naukova Dumka Kiev, 1977; English transl., Birkhäuser, 1986.
- [2] Levitan B.M., Inverse Sturm-Liouville problems. Nauka, Moscow, 1984; English transl., VNU Sci.Press, Utrecht, 1987.
- [3] Freiling G. and Yurko V.A., Inverse Sturm-Liouville Problems and their Applications. NOVA Science Publishers, New York, 2001.
- [4] Beals R., Deift P. and Tomei C., Direct and Inverse Scattering on the Line, Math. Surveys and Monographs, v.28. Amer. Math. Soc. Providence: RI, 1988.
- [5] Yurko V.A. Method of Spectral Mappings in the Inverse Problem Theory. Inverse and Ill-posed Problems Series. VSP, Utrecht, 2002.
- [6] Yurko V.A. Inverse Spectral Problems for Differential Operators and their Applications. Gordon and Breach, Amsterdam, 2000.
- [7] Lakshmikantham V. and Rama Mohana Rao M. Theory of integro-differential equations. Stability and Control: Theory and Applications, vol.1, Gordon and Breach, Singapore, 1995.
- [8] Yurko V.A., An inverse problem for integro-differential operators. Matem. zametki, 50, no.5 (1991), 134-146 (Russian); English transl. in Math. Notes, 50, no.5-6 (1991), 1188-1197.
- [9] Kuryshova Yu. An inverse spectral problem for differential operators with integral delay. Tamkang J. Math. 42, no.3 (2011), 295-303.
- [10] Buterin S.A. On the reconstruction of the convolution perturbation of the Sturm-Liouville operator from the spectrum, Differential Equations 46, no.1 (2010), 150–154.
- [11] Hromov A.P. On generating functions of Volterra operators. Math. USSR Sbornik 31, no.3 (1997), 409-432.